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1998 J. Phys. A: Math. Gen. 31 6517

(http://iopscience.iop.org/0305-4470/31/30/016)

J. Phys. A: Math. Gen. **31** (1998) 6517–6523. Printed in the UK PII: S0305-4470(98)90287-6

Canonical gauge equivalences of the sAKNS and sTB hierarchies

Jiin-Chang Shaw†§ and Ming-Hsien Tu‡‖

† Department of Applied Mathematics, National Chiao Tung University, Hsinchu, Taiwan
‡ Department of Physics, National Tsing Hua University, Hsinchu, Taiwan

Received 5 January 1998, in final form 18 May 1998

Abstract. We study the gauge transformations between the supersymmetric AKNS (sAKNS) and supersymmetric two-boson (sTB) hierarchies. The Hamiltonian nature of these gauge transformations is investigated, which turns out to be canonical. We also obtain the Darboux–Bäcklund transformations for the sAKNS hierarchy from these gauge transformations.

1. Introduction

During the past ten years, the theory of the soliton [1–3] has played an important role in theoretical and mathematical physics, especially in the explorations of the relationship between integrable models and string theories [4]. On the one hand, several kinds of correlation functions in string theory are governed by the integrable hierarchy equations (e.g. Korteweg–de Vries (KdV), Kadomtsev–Petviashvili (KP) etc) [4]. On the other hand, the idea of the supersymmetric extensions of the integrable systems [5–7] has motivated people to use them to study the theory of superstrings [8].

Recently, several supersymmetric integrable systems have been proposed and studied (see, e.g., [9–17] and references therein). In this paper, we discuss only two of them; the supersymmetric Ablowitz–Kaup–Newell–Segur (sAKNS) hierarchy [13] and the supersymmetric two-boson (sTB) hierarchy [11]. The former was introduced from the study of the reduction scheme in the constrained KP hierarchy [18], and the latter was constructed from the supersymmetric extension of the dispersive long water wave equation [19, 20]. Both of them have supersymmetric Lax representations, being bi-Hamiltonian, and have infinite conserved quantities etc. Besides these properties, these two hierarchies can be related to each other via a gauge transformation [13]. Sometimes, such transformation from one hierarchy to the other is called Miura transformation. However, from our viewpoint, the connection between these two hierarchies has not been totally explored. The purpose of this work is to provide a deeper understanding about the gauge transformations between the sAKNS and the sTB hierarchies.

Our paper is organized as follows: in section 2, we recall the Lax formulation of the sAKNS hierarchy. We then discuss the gauge transformations between the sAKNS and the sTB hierarchies. Section 3 is devoted to the investigation of the canonical property of these gauge transformations from the bi-Hamiltonian viewpoint. Our approach

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§ E-mail address: shaw@math.nctu.edu.tw
‖ E-mail address: mhtu@phys.nthu.edu.tw

follows very closely that of [21, 22] for other systems. We then show, in section 4, that the Darboux–Bäcklund transformations (DBTs) for the sAKNS hierarchy itself can be constructed from these gauge transformations. Concluding remarks are presented in section 5.

2. sAKNS and sTB hierarchies

The sAKNS hierarchy [13] has the Lax operator of the form

$$L = \partial + \Phi D^{-1} \Psi \tag{2.1}$$

which satisfies the hierarchy equations

$$\frac{\partial L}{\partial t_n} = [L^n_+, L] \tag{2.2}$$

where $D = \partial_\theta + \theta\partial$ is the supercovariant derivative defined on a (1|1) superspace [23] with coordinates (x, θ). $D^{-1} = \theta + \partial_\theta \partial^{-1}$ is the formal inverse of D, which satisfies $D^{-1}D = D^{-1}D = 1$. The multiplication rule for D acting on an arbitrary superfield U is $DU = (DU) + (-1)^{|U|} U D$. Here, we refer to the parity of a superfield U to be even if $|U| = 0$ and odd if $|U| = 1$. The coefficients functions Φ and Ψ are superfields with proper parity such that L is a bosonic operator. It can be proved that (2.2) is consistent with the following equations

$$\frac{\partial \Phi}{\partial t_n} = (L^n_+ \Phi) \qquad \frac{\partial \Psi}{\partial t_n} = -((L^n)^*_+ \Psi) \tag{2.3}$$

where the conjugate operation '$*$' is defined by $(AB)^* = (-1)^{|A||B|} B^* A^*$ for the super-pseudo-differential operators A, B and $f^* = f$ for the arbitrary superfield f. Therefore, Φ and Ψ are the eigenfunction and adjoint eigenfunction of the hierarchy, respectively. It can be shown [13] that the hierarchy equations (2.2) are invariant under the supersymmetric transformations: $\delta_\epsilon \Phi = \epsilon(D^\dagger \Phi)$ and $\delta_\epsilon \Psi = \epsilon(D^\dagger \Psi)$ where ϵ is an odd constant and $D^\dagger \equiv \partial_\theta - \theta\partial$.

Since the Lax operator (2.1) is assumed to be homogeneous under Z_2-grading, the gradings of the (adjoint) eigenfunction should satisfy $|\Phi| + |\Psi| = 1$. There are two cases to be discussed:

(a) $|\Phi| = 0$ and $|\Psi| = 1$,
(b) $|\Phi| = 1$ and $|\Psi| = 0$.

In the following, the sAKNS Lax operators for the case (a) and case (b) will be denoted by $L_\mathrm{a} = \partial + \Phi_\mathrm{a} D^{-1} \Psi_\mathrm{a}$ and $L_\mathrm{b} = \partial + \Phi_\mathrm{b} D^{-1} \Psi_\mathrm{b}$, respectively, and thus $|\Phi_\mathrm{a}| = |\Psi_\mathrm{b}| = 0$ and $|\Psi_\mathrm{a}| = |\Phi_\mathrm{b}| = 1$. For both cases, (2.2) contains the ordinary AKNS hierarchy equations in the bosonic limit.

Given a sAKNS hierarchy we can construct a non-standard Lax hierarchy via a gauge transformation. For case (a), let us perform the following transformation

$$M_\mathrm{a} : L_\mathrm{a} \to K = \Phi_\mathrm{a}^{-1} L_\mathrm{a} \Phi_\mathrm{a} \equiv \partial - (DJ_0) + D^{-1} J_1 \tag{2.4}$$

where both J_0 and J_1 are odd superfields which can be expressed in terms of Φ_a and Ψ_a as follows

$$J_0 = -(D \ln \Phi_\mathrm{a}) \qquad J_1 = \Phi_\mathrm{a} \Psi_\mathrm{a}. \tag{2.5}$$

The hierarchy equations then become

$$\frac{\partial K}{\partial t_n} = [K^n_{\geqslant 1}, K] \tag{2.6}$$

which is the so-called sTB hierarchy [11]. It can be shown [11] that the hierarchy equations (2.6) are invariant under the supersymmetric transformations: $\delta_\epsilon J_0 = \epsilon(D^\dagger J_0)$, $\delta_\epsilon J_1 = \epsilon(D^\dagger J_1)$.

For case (b), we need another gauge transformation to do the job since $|\Phi_{\rm b}| = 1$ in this case. Let us consider the following transformation

$$M_{\rm b} : L_{\rm b} \to K = D^{-1}\Psi_{\rm b} L_{\rm b} \Psi_{\rm b}^{-1} D \equiv \partial - (DJ_0) + D^{-1} J_1 \tag{2.7}$$

which implies that

$$J_0 = (D \ln \Psi_{\rm b}) \qquad J_1 = \Phi_{\rm b}\Psi_{\rm b} + (D^3 \ln \Psi_{\rm b}) \tag{2.8}$$

and the Lax operator K still satisfies the hierarchy equations (2.6).

In fact, both gauge transformations $M_{\rm a}$ and $M_{\rm b}$ have their inverse transformations $N_{\rm a}$ and $N_{\rm b}$, respectively. In other words, for a given sTB Lax operator K, one can perform the following transformation to gauge away the constant term and to obtain the Lax operator $L_{\rm a}$ [13]

$$N_{\rm a} : K \to L_{\rm a} = {\rm e}^{-\int^x (DJ_0)} K\, {\rm e}^{\int^x (DJ_0)} \equiv \partial + \Phi_{\rm a} D^{-1} \Psi_{\rm a} \tag{2.9}$$

where

$$\Phi_{\rm a} = {\rm e}^{-\int^x (DJ_0)} \qquad \Psi_{\rm a} = J_1\, {\rm e}^{\int^x (DJ_0)}. \tag{2.10}$$

It can be proved that $L_{\rm a}$ satisfies (2.2) if K satisfies (2.6).

Similarly, for case (b), we have

$$N_{\rm b} : K \to L_{\rm b} = {\rm e}^{-\int^x (DJ_0)} DKD^{-1}\, {\rm e}^{\int^x (DJ_0)} \equiv \partial + \Phi_{\rm b} D^{-1} \Psi_{\rm b} \tag{2.11}$$

where

$$\Phi_{\rm b} = (J_1 - J_{0x})\, {\rm e}^{-\int^x (DJ_0)} \qquad \Psi_{\rm b} = {\rm e}^{\int^x (DJ_0)}. \tag{2.12}$$

We would like to mention that the parity of the gauge operator associated with the gauge transformation $M_{\rm a}$ is even, whereas for $M_{\rm b}$ is odd. Since $N_{\rm a}(N_{\rm b})$ is the inverse of $M_{\rm a}$ ($M_{\rm b}$) and *vice versa*, thus we obtain the correspondences between the sAKNS and sTB hierarchies.

3. Canonical property and Hamiltonian structures

The discussions presented in the previous section establish the gauge equivalences between the sAKNS and the sTB hierarchies at Lax formulation level. In this section, we would like to discuss the Hamiltonian nature of these gauge transformations. Let us start from the sTB hierarchy.

The Lax equation (2.6) of the sTB hierarchy has a bi-Hamiltonian description as follows

$$\partial_{t_n} \begin{pmatrix} J_0 \\ J_1 \end{pmatrix} = \Theta_1 \begin{pmatrix} \delta H_{n+1}/\delta J_0 \\ \delta H_{n+1}/\delta J_1 \end{pmatrix} = \Theta_2 \begin{pmatrix} \delta H_n/\delta J_0 \\ \delta H_n/\delta J_1 \end{pmatrix} \tag{3.1}$$

where the first structure Θ_1 and the second structure Θ_2 are given by [11]

$$\Theta_1 = \begin{pmatrix} 0 & -D \\ -D & 0 \end{pmatrix} \tag{3.2}$$

$$\Theta_2 = \begin{pmatrix} 2D + 2D^{-1}J_1D^{-1} - D^{-1}J_{0x}D^{-1} & -D^3 + D(DJ_0) - D^{-1}J_1 D \\ D^3 + (DJ_0)D + DJ_1D^{-1} & J_1D^2 + D^2 J_1 \end{pmatrix} \tag{3.3}$$

which have been investigated [11] and found to be compatible by using the prolongation method [24]. The Hamiltonians H_n are defined by

$$H_n = \frac{-1}{n}\mathrm{str}K^n \equiv \frac{-1}{n}\int \mathrm{d}x\,\mathrm{d}\theta\,\mathrm{sres}K^n \tag{3.4}$$

where the super-residue (sres) picks up the coefficient of the D^{-1} term of a super-pseudo-differential operator.

Since the bi-Hamiltonian structure is one of the most important properties of an integrable system, it is quite natural to ask whether the gauge transformations discussed here are canonical or not. To see this, from the gauge transformation N_a, we can obtain the linearized map N'_a and its transposed map $N'^{\dagger}_\mathrm{a}$ as follows

$$N'_\mathrm{a} = \begin{pmatrix} -\Phi_\mathrm{a}D^{-1} & 0 \\ \Psi_\mathrm{a}D^{-1} & \Phi_\mathrm{a}^{-1} \end{pmatrix} \qquad N'^{\dagger}_\mathrm{a} = \begin{pmatrix} D^{-1}\Phi_\mathrm{a} & -D^{-1}\Psi_\mathrm{a} \\ 0 & \Phi_\mathrm{a}^{-1} \end{pmatrix} \tag{3.5}$$

where Φ_a and Ψ_a are related to J_0 and J_1 via equation (2.5) (or equation (2.10)). A straightforward calculation shows that

$$N'_\mathrm{a}\Theta_1 N'^{\dagger}_\mathrm{a} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv P_\mathrm{a} \tag{3.6}$$

$$\begin{aligned} N'_\mathrm{a}\Theta_2 N'^{\dagger}_\mathrm{a} &= \begin{pmatrix} \begin{matrix} -\Phi_\mathrm{a}D^{-2}\Phi_\mathrm{a}D - D\Phi_\mathrm{a}D^{-2}\Phi_\mathrm{a} \\ -2\Phi_\mathrm{a}D^{-2}\Phi_\mathrm{a}\Psi_\mathrm{a}D^{-2}\Phi_\mathrm{a} \end{matrix} & \begin{matrix} D^2 + D\Phi_\mathrm{a}D^{-2}\Psi_\mathrm{a} + \Phi_\mathrm{a}D^{-2}(D\Psi_\mathrm{a}) \\ +2\Phi_\mathrm{a}D^{-2}\Phi_\mathrm{a}\Psi_\mathrm{a}D^{-2}\Psi_\mathrm{a} \end{matrix} \\ \begin{matrix} D^2 + \Psi_\mathrm{a}D^{-2}\Phi_\mathrm{a}D + (D\Psi_\mathrm{a})D^{-2}\Phi_\mathrm{a} \\ +2\Psi_\mathrm{a}D^{-2}\Phi_\mathrm{a}\Psi_\mathrm{a}D^{-2}\Phi_\mathrm{a} \end{matrix} & \begin{matrix} -\Psi_\mathrm{a}D^{-2}(D\Psi_\mathrm{a}) - (D\Psi_\mathrm{a})D^{-2}\Psi_\mathrm{a} \\ -2\Psi_\mathrm{a}D^{-2}\Phi_\mathrm{a}\Psi_\mathrm{a}D^{-2}\Psi_\mathrm{a} \end{matrix} \end{pmatrix} \\ &\equiv Q_\mathrm{a} \end{aligned} \tag{3.7}$$

where P_a and Q_a are just the first and the second Hamiltonian structures obtained in [14]. Moreover, it has been shown [14] that P_a and Q_a are compatible through the method of prolongation and describe the hierarchy equations (2.2) as follows

$$\partial_{t_n}\begin{pmatrix} \Phi_\mathrm{a} \\ \Psi_\mathrm{a} \end{pmatrix} = P_\mathrm{a}\begin{pmatrix} \delta H_{n+1}/\delta\Phi_\mathrm{a} \\ \delta H_{n+1}/\delta\Psi_\mathrm{a} \end{pmatrix} = Q_\mathrm{a}\begin{pmatrix} \delta H_n/\delta\Phi_\mathrm{a} \\ \delta H_n/\delta\Psi_\mathrm{a} \end{pmatrix} \tag{3.8}$$

where the Hamiltonians H_n are defined by $H_n = -(1/n)\mathrm{str}L_\mathrm{a}^n$. Hence, the gauge transformation N_a (or M_a) is a canonical map.

Next, let us turn to the gauge transformation N_b. From (2.11), the linearized map N'_b and its transposed map $N'^{\dagger}_\mathrm{b}$ can be constructed as follows

$$N'_\mathrm{b} = \begin{pmatrix} -\Phi_\mathrm{b}D^{-1} - \Psi_\mathrm{b}^{-1}\partial & \Psi_\mathrm{b}^{-1} \\ \Psi_\mathrm{b}D^{-1} & 0 \end{pmatrix} \qquad N'^{\dagger}_\mathrm{b} = \begin{pmatrix} \partial\Psi_\mathrm{b}^{-1} + D^{-1}\Phi_\mathrm{b} & -D^{-1}\Psi_\mathrm{b} \\ \Psi_\mathrm{b}^{-1} & 0 \end{pmatrix} \tag{3.9}$$

where Φ_b and Ψ_b are related to J_0 and J_1 via (2.8) (or (2.12)). Using (3.9), we can obtain two Poisson structures of the sAKNS hierarchy for the case (b). After some algebras, we have

$$N'_\mathrm{b}\Theta_1 N'^{\dagger}_\mathrm{b} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv -P_\mathrm{b} \tag{3.10}$$

$$\begin{aligned} N'_\mathrm{b}\Theta_2 N'^{\dagger}_\mathrm{b} &= \begin{pmatrix} \begin{matrix} -\Phi_\mathrm{b}D^{-2}(D\Phi_\mathrm{b}) - (D\Phi_\mathrm{b})D^{-2}\Phi_\mathrm{b} \\ -2\Phi_\mathrm{b}D^{-2}\Phi_\mathrm{b}\Psi_\mathrm{b}D^{-2}\Phi_\mathrm{b} \end{matrix} & \begin{matrix} D^2 + \Phi_\mathrm{b}D^{-2}\Psi_\mathrm{b}D + (D\Phi_\mathrm{b})D^{-2}\Psi_\mathrm{b} \\ +2\Phi_\mathrm{b}D^{-2}\Phi_\mathrm{b}\Psi_\mathrm{b}D^{-2}\Psi_\mathrm{b} \end{matrix} \\ \begin{matrix} D^2 + D\Psi_\mathrm{b}D^{-2}\Phi_\mathrm{b} + \Psi_\mathrm{b}D^{-2}(D\Phi_\mathrm{b}) \\ +2\Psi_\mathrm{b}D^{-2}\Phi_\mathrm{b}\Psi_\mathrm{b}D^{-2}\Phi_\mathrm{b} \end{matrix} & \begin{matrix} -\Psi_\mathrm{b}D^{-1}\Psi_\mathrm{b}D - D\Psi_\mathrm{b}D^{-2}\Psi_\mathrm{b} \\ -2\Psi_\mathrm{b}D^{-2}\Phi_\mathrm{b}\Psi_\mathrm{b}D^{-2}\Psi_\mathrm{b} \end{matrix} \end{pmatrix} \\ &\equiv -Q_\mathrm{b} \end{aligned} \tag{3.11}$$

which imply that the hierarchy equations (2.2) for case (b) can be written as

$$\partial_{t_n}\begin{pmatrix}\Phi_{\rm b}\\ \Psi_{\rm b}\end{pmatrix} = P_{\rm b}\begin{pmatrix}\delta H_{n+1}/\delta\Phi_{\rm b}\\ \delta H_{n+1}/\delta\Psi_{\rm b}\end{pmatrix} = Q_{\rm b}\begin{pmatrix}\delta H_{n}/\delta\Phi_{\rm b}\\ \delta H_{n}/\delta\Psi_{\rm b}\end{pmatrix}. \tag{3.12}$$

Note that the parity of the gauge operator of the gauge transformation $N_{\rm b}$ is odd. Hence, from (3.4), (2.11) and the identity $\mathrm{str}AB = (-1)^{|A||B|}\mathrm{str}BA$, the Hamiltonians in (3.12) can be expressed in terms of $L_{\rm b}$ as $(1/n)\mathrm{str}L_{\rm b}^n$ which are just the Hamiltonians of the sAKNS hierarchy defined earlier with a minus sign. Therefore, the minus sign appearing in the front of $P_{\rm b}$ and $Q_{\rm b}$ in (3.10) and (3.11) is used to compensate the sign from the Hamiltonians. We follow the same line in [14] to investigate the Jacobi identity for $P_{\rm b}$ and $Q_{\rm b}$ by using the prolongation method. It turns out that $P_{\rm b}$ and $Q_{\rm b}$ are compatible and indeed define a bi-Hamiltonian structure of the associated hierarchy. Hence, just like $N_{\rm a}$, the gauge transformation $N_{\rm b}$ is canonical as well.

To sum up, the canonical property of the gauge transformations between the sAKNS and sTB hierarchies can be summarized as follows

$$N_i'\Theta_1 N_i'^{\dagger} = (-1)^{|N_i|}P_i \qquad N_i'\Theta_2 N_i'^{\dagger} = (-1)^{|N_i|}Q_i \qquad i = a, b. \tag{3.13}$$

4. Darboux–Bäcklund transformations

Having constructed the canonical gauge transformations between the sAKNS and sTB hierarchies, now we would like to use these gauge transformations to derive the Darboux–Bäcklund transformations (DBTs) for the sAKNS hierarchy itself. Given a sAKNS Lax operator, say $L_{\rm a}$, we can perform the gauge transformation $M_{\rm a}$ followed by $N_{\rm b}$ to obtain the Lax operator $L_{\rm b}$ as follows

$$L_{\rm a} \overset{M_{\rm a}}{\rightarrow} K \overset{N_{\rm b}}{\rightarrow} L_{\rm b}. \tag{4.1}$$

That is, using (2.4) and (2.11), we can define the gauge operator $T(\Phi_{\rm a}) = \Phi_{\rm a}D\Phi_{\rm a}^{-1}$ such that

$$L_{\rm a} \rightarrow L_{\rm b} = TL_{\rm a}T^{-1} \equiv \partial + \Phi_{\rm b}D^{-1}\Psi_{\rm b} \tag{4.2}$$

where the (adjoint) eigenfunctions are related by

$$\Phi_{\rm b} = \Phi_{\rm a}(\Phi_{\rm a}\Psi_{\rm a} + (D^3\ln\Phi_{\rm a})) \tag{4.3}$$

$$\Psi_{\rm b} = \Phi_{\rm a}^{-1}. \tag{4.4}$$

Notice that although the gauge transformation (4.2) preserves the form of the Lax operator and the Lax formulations, the parity of the transformed (adjoint) eigenfunction has been changed due to the fact that the parity of the gauge operator T is odd. Thus, strictly speaking, the gauge transformation (4.2) is not a DBT but a 'quasi-DBT'. On the other hand, we can construct another quasi-DBT from $L_{\rm b}$ to $L_{\rm a}$ as follows

$$L_{\rm b} \overset{M_{\rm b}}{\rightarrow} K \overset{N_{\rm a}}{\rightarrow} L_{\rm a} \tag{4.5}$$

which is triggered by the gauge operator $S(\Psi_{\rm b}) = \Psi_{\rm b}^{-1}D^{-1}\Psi_{\rm b}$ such that

$$L_{\rm b} \rightarrow L_{\rm a} = SL_{\rm b}S^{-1} \equiv \partial + \Phi_{\rm a}D^{-1}\Psi_{\rm a}. \tag{4.6}$$

Here

$$\Phi_{\rm a} = \Psi_{\rm b}^{-1} \tag{4.7}$$

$$\Psi_{\rm a} = \Phi_{\rm b}(\Phi_{\rm b}\Psi_{\rm b} + (D^3\ln\Psi_{\rm b})). \tag{4.8}$$

Note that both quasi-DBTs (4.2) and (4.6) are canonical since they are constructed out from the canonical transformations M_i and N_i. We also remark that the form of the gauge operator T was first considered in [25] for studying the DBT for the Manin–Radul super KdV equation [5]. Motivated by the above discussions, we may have true DBTs by considering the hierarchy equations (2.2) associated with the Lax operator

$$L = \partial + \Phi_1 D^{-1}\Psi_1 + \Phi_2 D^{-1}\Psi_2 \tag{4.9}$$

with parity $|\Phi_1| = |\Psi_2| = 0$ and $|\Psi_1| = |\Phi_2| = 1$. Let us consider the DBT triggered by the eigenfunction Φ_1 as follows

$$\begin{aligned} L \to \hat{L} &= TLT^{-1} \qquad T(\Phi_1) \equiv \Phi_1 D \Phi_1^{-1} \\ &\equiv \partial + \hat{\Phi}_1 D^{-1}\hat{\Psi}_1 + \hat{\Phi}_2 D^{-1}\hat{\Psi}_2 \end{aligned} \tag{4.10}$$

where the transformed (adjoint) eigenfunctions are given by

$$\begin{aligned} \hat{\Phi}_1 &= \Phi_1(D\Phi_1^{-1}\Phi_2) = (T(\Phi_1)\Phi_2) \\ \hat{\Psi}_1 &= \Phi_1^{-1}(D^{-1}\Phi_1\Psi_2) = (S(\Phi_1)\Psi_2) \\ \hat{\Phi}_2 &= \Phi_1(\Phi_1\Psi_1 - \Phi_2\Psi_2 + (D^3 \ln \Phi_1) + (D\Phi_1^{-1}\Phi_2)(D^{-1}\Phi_1\Psi_2)) = (T(\Phi_1)L\Phi_1) \\ \hat{\Psi}_2 &= \Phi_1^{-1} \end{aligned} \tag{4.11}$$

with parity $|\hat{\Phi}_1| = |\hat{\Psi}_2| = 0$ and $|\hat{\Psi}_1| = |\hat{\Phi}_2| = 1$. On the other hand, we can consider the DBT triggered by the adjoint eigenfunction Ψ_2 as follows

$$L \to \hat{L} = SLS^{-1} \qquad S(\Psi_2) \equiv \Psi_2^{-1} D^{-1} \Psi_2 \equiv \partial + \hat{\Phi}_1 D^{-1}\hat{\Psi}_1 + \hat{\Phi}_2 D^{-1}\hat{\Psi}_2 \tag{4.12}$$

where

$$\begin{aligned} \hat{\Phi}_1 &= \Psi_2^{-1} \\ \hat{\Psi}_1 &= (\Phi_2\Psi_2 - \Phi_1\Psi_1 + (D^3 \ln \Psi_2) + (D^{-1}\Psi_2\Phi_1)(D\Psi_1\Psi_2^{-1}))\Psi_2 = -(T(\Psi_2)L^*\Psi_2) \\ \hat{\Phi}_2 &= \Psi_2^{-1}(D^{-1}\Psi_2\Phi_1) = (S(\Psi_2)\Phi_1) \\ \hat{\Psi}_2 &= \Psi_2(D\Psi_2^{-1}\Psi_1) = (T(\Psi_2)\Psi_1) \end{aligned} \tag{4.13}$$

with parity $|\hat{\Phi}_1| = |\hat{\Psi}_2| = 0$ and $|\hat{\Psi}_1| = |\hat{\Phi}_2| = 1$. Finally, we would like to mention that the above scheme can be generalized to a class of supersymmetric hierarchies which have Lax operators of the form

$$L = \partial + \sum_{i=1}^{n}(\Phi_{2i-1} D^{-1}\Psi_{2i-1} + \Phi_{2i} D^{-1}\Psi_{2i}) \qquad (n \geqslant 1) \tag{4.14}$$

with parity $|\Phi_{2i-1}| = |\Psi_{2i}| = 0$ and $|\Phi_{2i}| = |\Psi_{2i-1}| = 1$. The gauge operators of the DBTs then can be constructed from the even (adjoint) eigenfunctions as $T_i = \Phi_{2i-1} D \Phi_{2i-1}^{-1}$ or $S_i = \Psi_{2i}^{-1} D^{-1} \Psi_{2i}$ which not only preserve the Lax formulations but also the parity content of the (adjoint) eigenfunctions in the Lax operator.

5. Concluding remarks

We have established the gauge equivalences between the sAKNS and sTB hierarchies. We have also shown that the gauge transformations connecting these two hierarchies are canonical, in the sense that the bi-Hamiltonian structure of the sAKNS hierarchy is mapped to the bi-Hamiltonian structure of the sTB hierarchy according to equation (3.13). Using these gauge transformations, the (quasi) DBTs for the sAKNS hierarchy and its generalizations can be constructed, which turns out to be canonical as well. Some other topics such as iterated DBTs, soliton solutions and non-local conserved charges of these hierarchies are worth further investigation [26]. We leave this work to a future publication.

Acknowledgments

We would like to thank Professor W-J Huang for helpful discussions. This work was supported by the National Science Council of Taiwan under grant No NSC-87-2811-M-007-0025. MHT also wishes to thank the National Centre for Theoretical Sciences of the National Science Council of Taiwan for partial support.

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